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Field constructions and their composition limits

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Abstract: This note retains five scoped investigations accompanying *Information clustering and geometric pressure in AdS_3*: a complementary-spectrum obstruction, an interval-entropy construction and its composition limit, exact transport with supplied conformal probes, density-matrix geometry, and replica channel weights. Each calculation specifies its own field data and admissible operations. They are separate tasks and do not upgrade the main paper's actual-state entropy saving into an exact finite-wall identity or a general no-go theorem. References to the main paper use the equation and section numbering of the main PDF supplied in the same package.

Keywords: Quantum Entanglement, Conformal Field Theory, AdS-CFT Correspondence

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The main paper owns the definitions of G_τ, the positive exact-trace pressure problem, its calibration and the ultraviolet correspondence. In this note, an external section or equation reference means that main paper; local numbered sections contain the retained alternative-task calculations. The complete numerical producers remain in the common source package. These calculations are preserved as evidence with their original hypotheses, rather than used as additional proofs of the main task.

1 Why the full spectrum is insufficient

A function of the union spectrum alone discards the spatial component data used by G_τ. Equal-dimensional complementary reductions provide a direct test: their full spectra coincide, but the positive-area source distinguishes their homology regions. Consider the time-symmetric vacuum disk

$$\mathrm{d}s^2 = R^2(\mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\phi^2), \qquad 0 \le \rho \le \rho_c, \tag{1.1}$$

with anchors on the cutoff circle. Take

$$A = [0, \pi/6] \cup [2\pi/3, 3\pi/2], \qquad B = A^c. \tag{1.2}$$

Both have total angular length π, two boundary components and the same four endpoints. On a uniform pure-state lattice regulator with $12m$ equal local factors, both contain $6m$ factors. Their reduced density operators therefore have identical full spectra, including zero multiplicities: the two reductions of a pure bipartite state are the two square products of its coefficient matrix.

The positive homology areas differ. Put $C = \cosh\rho_c$ and $S = \sinh\rho_c$. A geodesic chord closing a minor boundary arc of angle θ has length and cap area

$$\begin{aligned} \ell(\theta) &= 2R\,\mathrm{arsinh}\left(S\sin\frac{\theta}{2}\right), \\ v(\theta) &= R^2[\theta C - \pi + 2\beta_\theta], \qquad \beta_\theta = \arctan\frac{\cot(\theta/2)}{C}. \end{aligned} \tag{1.3}$$

The cap follows by subtracting the geodesic triangle from the radial sector, using its angles $\theta, \beta_\theta, \beta_\theta$. Concavity of $\operatorname{arsinh}(S\sqrt{x})$ and $\sin^2(\pi/12) + \sin^2(5\pi/12) = 1$ give

$$L_D := \ell(\pi/6) + \ell(5\pi/6) < L_C := 2\ell(\pi/2). \tag{1.4}$$

The unique zero-source free boundary is therefore the same pair of chords for A and B. The positive side is two caps for A and their connected complement for B.

Writing $b = \beta_{\pi/6} + \beta_{5\pi/6}$ gives

$$\begin{aligned} V_A &= R^2(\pi C - 2\pi + 2b), & V_B &= R^2(\pi C - 2b), \\ V_B - V_A &= 2\pi R^2 - 4R^2 \arctan \frac{4C}{S^2} > 0. \end{aligned} \tag{1.5}$$

The difference approaches $2\pi R^2$ at the boundary. Its topological origin is consistent with the Gauss–Bonnet volume analysis of [1].

The distinction persists after optimizing at a strictly positive source. For this separate disk witness, X denotes the prescribed interior trace of a regular compact positive region $\Omega = \overline{\operatorname{int}\Omega}$, and L_{free} counts only its free boundary. Zero-width cracks and appended one-dimensional pieces are excluded. Define $\mathcal{F}_X(s) = \min_{\operatorname{tr}_{\partial D}\mathbf{1}_\Omega = \mathbf{1}_X}[L_{\text{free}} + sV]$ without the overall $1/(4G)$. Complementation gives

$$\mathcal{F}_B(s) = sV_D + \mathcal{F}_A(-s), \qquad V_D = 2\pi R^2(C-1). \tag{1.6}$$

For a chord of zero-source length $2Ra$, set $k = sR$ and $q_k = \sqrt{1-k^2}$. The calibrated pressure arc has

$$L_k = \frac{2R}{q_k} \operatorname{arsinh}(q_k \sinh a), \qquad W(k) = R^2 \left[\frac{kL_k}{R} - 2\arcsin(k \tanh a) \right]. \tag{1.7}$$

Here $W(k)$ is the signed area removed from the original cap; it is odd in k. The optimized cap action is

$$2R\,[q_k \operatorname{arsinh}(q_k \sinh a) + k \arcsin(k \tanh a)] + s\,v_0. \tag{1.8}$$

The bracket is even in k, so on the common stable pairing

$$\mathcal{F}_B(s) - \mathcal{F}_A(s) = s(V_B - V_A). \tag{1.9}$$

The two optimized free boundaries at nonzero source are displaced differently; the argument is not an evaluation on fixed RT chords. The calibrated arcs also remain inside the disk. If their dimensional radial coordinate r had an interior maximum above the anchor radius $R\rho_c$, the unit-speed radial equation and curvature bound would give

$$r'' \geq \frac{1}{R} \coth(r/R) - |s| > 0, \qquad |s|R < 1, \tag{1.10}$$

contradicting that maximum. Thus the unconstrained single-arc certificate applies to the disk-constrained problem as well.

For $R = 1$, $\rho_c = 2$ and $s = 0.01$, the explicit global values are

$$\mathcal{F}_A = 5.680250966746210, \qquad \mathcal{F}_B = 5.708983731193190. \tag{1.11}$$

The pairing gap is $L_C - L_D = 1.07584868449$, while $sV_D = 0.17355387382$, so the other pairing cannot win. The original supporting geodesics are separated by 1.92207837705; the two pressure displacements are bounded in sum by $2\,\mathrm{artanh}(0.01) = 0.02000066671$, ensuring disjointness. The single-arc calibration and removal of closed loops then establish these as attainable global minima.

There is also a fixed-source continuum witness. As $\rho_c \to \infty$, the alternative-pairing action gaps for A and B approach, respectively,

$$2R\left[\sqrt{1-k^2}\log 2 + \pi k\right], \qquad 2R\left[\sqrt{1-k^2}\log 2 - \pi k\right]. \tag{1.12}$$

Both are positive for

$$0 < k < \frac{\log 2}{\sqrt{\pi^2 + \log^2 2}}. \tag{1.13}$$

The supporting-geodesic separation tends to a positive value larger than twice the displacement bound throughout this interval, so the common pairing remains attainable. Equation (1.9) then gives

$$\lim_{\rho_c \to \infty} \frac{\mathcal{F}_B(s) - \mathcal{F}_A(s)}{4G} = \frac{2\pi R^2 s}{4G} = \frac{\pi c k}{3}. \tag{1.14}$$

Thus the discrepancy is finite in the cutoff and leading in c, rather than an effect requiring an exact microscopic identification at a finite wall.

Any arbitrary-unitary-invariant functional of the reduced density matrix must assign the same value to the two equal-dimensional complementary spectra, in conflict with (1.9). The obstruction is consequently to a spectral candidate for the stated positive-area target. It does not apply to a state equipped with local algebras, a preferred tensor decomposition, a reference or other spatial structure.

The regulator and subtraction qualifications are important here. The lattice establishes the exact spectral identity; the radial wall defines the exact geometric formulas. An exact microscopic equivalence of those finite regulators has not been supplied. The continuum finite difference also survives common boundary-only subtractions, because the regions have equal cutoff length and identical endpoints. A finite Euler-characteristic counterterm can change that difference, but also changes the geometric target and its source competition. Neither an arbitrary topology-dependent subtraction nor an exact finite-wall dictionary is assumed in this test.

2 An interval-entropy construction and its composition limit

Interval entropies provide enough local geometric data to reconstruct a single pressure arc. We give a variational realization with a finite buffer, explicit endpoint terms and a fixed reference, then test its direct composition on a disconnected union. The single-arc result is exact for that construction; the composition test identifies the outer-minus-gap sign that prevents it from replacing (1.3).

Fix the continuum vacuum entropy convention $S(u,v) = (c/3)\log[(v-u)/a_{\mathrm{UV}}] + s_0$, and two contact scales $b, \epsilon > 0$. The entropy regulator a_{UV} is distinct from the contact height ϵ. For a smooth family of intervals contained in $[-b-\epsilon, b+\epsilon]$, define

$$\begin{aligned} X &= (u+v)/2, \qquad r = (v-u)/2, \qquad p = r'/X', \qquad w = \sqrt{1-p^2}, \\ x &= X - rp, \qquad z = rw, \qquad D = X'(1-p^2) - rp'. \end{aligned} \tag{2.1}$$

The allowed families obey $u', v' > 0$, $D > 0$, $z \geq \epsilon$, and fixed contact endpoints $(x,z) = (-b,\epsilon), (b,\epsilon)$. In particular $x' = D > 0$. No final interval or endpoint slope is fixed independently of these conditions.

The information inputs are symmetric differential entropy and an integrated square-root conditional mutual information:

$$E_{\mathrm{sym}} = \frac{1}{2}\int (S_v v' - S_u u')\mathrm{d}t, \qquad K = \int \sqrt{\frac{12}{c} S_{uv} u' v'}\,\mathrm{d}t. \tag{2.2}$$

For adjacent monotone overlapping intervals, with intersection B_j and union U_j,

$$\begin{aligned} C_j &= S(I_j) + S(I_{j+1}) - S(B_j) - S(U_j) \geq 0, \\ E_{\mathrm{sym}} &= \frac{1}{2}\lim \sum_j [S(I_j) + S(I_{j+1}) - 2S(B_j)], \\ K &= \sqrt{12/c}\lim \sum_j \sqrt{C_j}. \end{aligned} \tag{2.3}$$

The limits here concern fixed nondegenerate smooth families. They do not exchange a lattice optimization with a continuum infimum. Differential entropy and kinematic CMI have established geometric interpretations [2–6]; those results do not identify the complete source functional below with an established communication or recovery cost.

Writing $[f] = f(1) - f(0)$, consider the fully specified relative-reference functional

$$\Phi_\lambda = E_{\mathrm{sym}} - \frac{c}{6}[\operatorname{artanh} p] + \frac{c\lambda}{6}\left[2\arctan\frac{b}{\epsilon} - K + [\arcsin p]\right]. \tag{2.4}$$

The endpoint completion and the reference are part of its definition. A direct calculation from (2.1) gives

$$\frac{6}{c}\Phi_\lambda = \int (\alpha - \lambda\beta) + 2\lambda \arctan(b/\epsilon). \tag{2.5}$$

For example, $E_{\mathrm{sym}} = (c/6)\int X'/r\,\mathrm{d}t$ and $K = \int X' w/r\,\mathrm{d}t$. Differentiating $x = X - rp$ and $z = rw$ yields $\sqrt{x'^2 + z'^2}/z = X'/r - p'/(1-p^2)$ and $x'/z = X'w/r - p'/w$, which proves both endpoint terms in (2.5). Thus this is an off-shell identity, rather than an evaluation on a chosen saddle.

For $0 < \lambda < 1$, the calibration of appendix D applies to every competitor. Set

$$\mathcal{R} = \frac{\lambda\epsilon + \sqrt{\epsilon^2 + q^2 b^2}}{q^2}, \qquad h = \lambda\mathcal{R}, \qquad y = \sqrt{\mathcal{R}^2 - x^2}. \tag{2.6}$$

The equality circle has a unique allowed interval-family lift, up to a change of increasing parameter,

$$X_*(x) = \frac{hx}{y}, \qquad r_*(x) = \mathcal{R}\left(1 - \frac{h}{y}\right), \qquad u_* = X_* - r_*, \quad v_* = X_* + r_*. \tag{2.7}$$

Indeed $p_* = -x/\mathcal{R}$, $X_*' = h\mathcal{R}^2/y^3 > 0$, $D_* = 1$, and $z_* = y - h \geq \epsilon$. Its support radius is

$$B_\lambda = \frac{\mathcal{R}(\epsilon + \lambda b)}{\epsilon + \lambda \mathcal{R}} = b + \epsilon \frac{\mathcal{R} - b}{\epsilon + \lambda \mathcal{R}} \leq b + \epsilon. \tag{2.8}$$

The last inequality follows from $(1-\lambda)\mathcal{R} \leq b+\epsilon$, obtained by squaring $\sqrt{\epsilon^2 + (1-\lambda^2)b^2} \leq \epsilon + (1+\lambda)b$. The fixed, source-independent buffer therefore contains the global minimizer. The all-path lower bound proves that restricting to it leaves the infimum unchanged.

Evaluation gives the exact relative-reference minimum

$$\inf \Phi_\lambda = \frac{c}{3} H_\lambda(b) = \frac{c}{3}\left[q \operatorname{arsinh} \frac{qb}{\epsilon} + \lambda \arcsin \frac{\lambda b}{\sqrt{b^2 + \epsilon^2}}\right]. \tag{2.9}$$

To obtain the actual positive cutoff-region action, the reference must instead be the cutoff chord:

$$\Phi_\lambda^{\mathrm{true}} = \Phi_\lambda + \frac{c\lambda}{6}\left[\frac{2b}{\epsilon} - 2 \arctan \frac{b}{\epsilon}\right], \qquad \inf \Phi_\lambda^{\mathrm{true}} = \frac{c}{6} f_{\lambda,\epsilon}(-b, b). \tag{2.10}$$

It has the same minimizing family. The two reference values must not be identified.

The same calibration controls stability and limits. In its Fermi coordinate t, a fixed-endpoint variation $\rho = -\operatorname{artanh} \lambda + \eta(t)$ has quadratic increment

$$\Delta\Phi_\lambda = \frac{cq}{12} \int [(\eta')^2 + \eta^2] \, \mathrm{d}t + O(\|\eta\|_{C^1}^3). \tag{2.11}$$

At $\lambda = 0$, (2.9) remains the infimum, but the strictly monotone interval-family class has only a limiting representation. The $\lambda > 0$ minimizing families, evaluated in Φ_0 and then taken to $\lambda \downarrow 0$, form an admissible sequence approaching its calibrated lower bound. The endpoint slopes tend to $\pm b/\sqrt{b^2 + \epsilon^2}$ while the interval derivatives collapse; discarding these contact data would incorrectly discard the endpoint contribution. At $\lambda = 1$, the interval task admits the horizontal family $X = x, r = \epsilon, p = 0$ and relative value $(c/3)\arctan(b/\epsilon)$. That path-domain endpoint does not assert attainment by a positive exact-trace BV region at critical source. The geometric theorem remains strictly subcritical.

For $u = b/\epsilon$, differentiating (2.9) gives

$$(\inf \Phi_\lambda)'' = -\frac{c}{3q^3}\left[\operatorname{arsinh}(qu) - \frac{qu}{\sqrt{1 + q^2u^2}}\right] < 0. \tag{2.12}$$

These source properties are not general channel or state-mixture monotonicity theorems. A local entropy-scheme change $S \mapsto S + f(u) + g(v)$ leaves K unchanged but shifts E_{sym} by $[g(v) - f(u)]/2$; a corresponding endpoint compensation is required to keep the same functional. The fixed-source UV expansion is

$$\inf \Phi_\lambda = \frac{c}{3}\left[q \log \frac{2b}{\epsilon} + q \log q + \lambda \arcsin \lambda\right] + O(c\epsilon^2/b^2). \tag{2.13}$$

Subtracting divergences, or changing the fixed reference, consequently changes the numerical interpretation and must be stated.

The construction does not compose by applying the same rule to the entropy of a disconnected union. Take translating windows $I_1(t) = [t, t + L]$ and $I_2(t) = [t + L + g, t + 2L + g]$, with $0 \le t \le T < g$. In their connected leading vacuum channel,

$$S_C = \frac{c}{3} \log \frac{(v_2 - u_1)(u_2 - v_1)}{a_{\mathrm{UV}}^2}, \qquad E_{\cup}^C = \frac{cT}{3} \left(\frac{1}{2L + g} - \frac{1}{g} \right) < 0. \tag{2.14}$$

The gap appears with a negative differential-entropy sign, although both boundaries of a positive homology region have positive length. Component contact slopes vanish, so the zero-source endpoint terms do not repair this sign. For $L = 1$, $g = 0.1$, $T = 0.02$, the contact anchors are $[-0.55, -0.53] \cup [0.55, 0.57]$ at $\epsilon = 0.5$. The union prescription divided by $c/6$ gives -0.380952380952381 at zero source; the positive-region minimum is 0.079994667626438. The buffers and the physical contact intervals are different regions, so their channel labels must not be conflated. This is a leading-channel counterexample to that prescribed composition, not a no-go for all algebra-aware information functionals. It explains why (1.3) treats actual block correlations directly rather than composing oriented differential entropies.

3 Exact finite-cutoff transport with supplied conformal probes

Supplied conformal probes define another field task, for which a finite-cutoff pressure correspondence is exact. Its inputs include independently prepared copies, separately metered controls and a tracked phase reference. We derive that result and examine how changing these inputs changes the optimization. The actual-state saving in the main text requires none of the supplied probe data.

3.1 Probe states and separately metered control

In a unitary CFT_2 with $c_L = c_R = c > 0$ and nonzero chiral stress tensor, the normalized state

$$|T\rangle = \sqrt{2/c}\, L_{-2}|0\rangle, \qquad L_1|T\rangle = 0, \qquad L_0|T\rangle = 2|T\rangle \tag{3.1}$$

is a global lowest-weight vector with $h = 2$. It is not a Virasoro primary. A positive-energy realization of a weight-h coherent orbit is

$$\psi_{x,z}(k) = \frac{(2z)^h}{\sqrt{\Gamma(2h)}} k^{h-1/2} e^{-zk + ixk}, \qquad k > 0, \quad z > 0. \tag{3.2}$$

The Gamma density $|\psi|^2$ has $\langle k \rangle = h/z$ and $\mathrm{Var}(k) = h/(2z^2)$. Since $\partial_x \psi = ik\psi$ and $\partial_z \psi = (h/z - k)\psi$, its moments and overlap give

$$\begin{aligned} \langle x, z | x', z' \rangle &= \left[\frac{2\sqrt{zz'}}{z + z' - i(x' - x)} \right]^{2h}, \\ \mathrm{d}s_{\mathrm{FS}}^2 &= \frac{h}{2z^2} (\mathrm{d}x^2 + \mathrm{d}z^2), \qquad \mathcal{A}_B = -h \frac{\mathrm{d}x}{z}, \qquad \langle P \rangle = h/z. \end{aligned} \tag{3.3}$$

At $h = 2$, the FS line element is the unit hyperbolic one and the Berry connection is -2β. These are actual normalized Hilbert-space data [7–9], not a fitted kernel.

Supply n independent copies of this probe. Preparation is outside the bill: global conformal controls cannot excite the invariant vacuum into $|T\rangle$. The other chiral sector remains in its vacuum. In the wavefunction representation $P = k$, $D = -i(k\partial_k + 1/2)$, and the self-adjoint affine controls

$$\frac{H_i(t)}{\hbar} = \left(-\dot{x}_i + x_i\frac{\dot{z}_i}{z_i}\right)P_i - \frac{\dot{z}_i}{z_i}D_i, \qquad \frac{\Delta H_i}{\hbar} = \frac{\sqrt{\dot{x}_i^2 + \dot{z}_i^2}}{z_i} \tag{3.4}$$

realize arbitrary smooth orbit paths. Impose the per-copy mean-P cap $\langle P_i\rangle \le 2/\epsilon$, equivalently $z_i \ge \epsilon$. It is neither a spectral cutoff nor a bound on the driving Hamiltonian, and the total endpoint expectation is $2n/\epsilon$.

The separately metered resource and the lifted fixed-width comparison phase are

$$\begin{aligned} C_{\text{add}} &= \int \sum_i \Delta H_i/\hbar \, \mathrm{d}t = \sum_i \int_{C_i} \alpha, \\ \Gamma &= 2\sum_i \left[\frac{v_{\pi(i)} - u_i}{\epsilon} - \int_{C_i}\beta\right]. \end{aligned} \tag{3.5}$$

The start rays are (u_i, ϵ); each terminal ray (v_j, ϵ) must be occupied once, with free classical choice of a bijection π. The target is a union of product-ray targets, not a coherent superposition. Entangling controls are excluded. The phase compares each trajectory with the oriented translation at fixed width and requires a real tracked record, beyond terminal holonomy modulo 2π. The source task is

$$\mathcal{J}_\lambda = \inf_{\pi,\{C_i\}}\left[C_{\text{add}} + \frac{\lambda}{2}\Gamma\right] = \frac{\lambda W}{\epsilon} + \min_\pi \sum_i d_\lambda^\epsilon(u_i, v_{\pi(i)}). \tag{3.6}$$

The last equality follows from $\sum_i (v_{\pi(i)} - u_i) = W$ and separate metering; no bulk pairing is imposed on the task.

3.2 The finite-cutoff correspondence

For $0 \le \lambda < 1$, the complete geometric proof of section 4 and appendix E applies directly to (3.6):

$$\mathcal{J}_\lambda = \min_{E\in\mathcal{A}_\epsilon(A)}[P_H(E; H_\epsilon) + \lambda\,\mathrm{Area}_H(E)], \qquad \frac{c}{6}\mathcal{J}_\lambda = F_\lambda^\epsilon. \tag{3.7}$$

The individual minimizing arcs are the exact pressure circles, free assignment excludes intersections by tail exchange, and endpoint parity reconstructs the positive region. Conversely, current decomposition bounds every exact-trace competitor by the same assignment. Thus (3.7) is exact at finite cutoff, for every fixed finite endpoint set. It does not obtain an exact finite-wall entropy identity for ρ_A.

The signed-source control statement extends to $|\lambda| < 1$. The arc calibration is valid for either sign, and

$$(1 - |\lambda|)\alpha \le \alpha - \lambda\beta \le (1 + |\lambda|)\alpha \tag{3.8}$$

replaces the nonnegative-source cycle bound. Every nonzero closed current has strictly positive directed action; the same equality classification therefore gives the signed positive-region minimum. Negative source values need not be nonnegative resources or entropies. This signed extension belongs to the supplied-probe task, not the nonnegative-tariff theorem for G_τ.

The normalized $h = 2$ orbit is independent of c. The factor $c/6$ in (3.7) is an entropy/gravity dictionary normalization using one copy's charge, not the charge nc of the auxiliary system. The controls are global and idealized; preparation, reference/readout, latency and extra driving constraints are not charged. Availability of this representation in a CFT does not imply that every such CFT has an Einstein dual.

Separate metering is also substantive. For a product path with independent controls,

$$(\Delta H_{\mathrm{tot}})^2 = \sum_i (\Delta H_i)^2, \qquad v_{\mathrm{FS,tot}} = \sqrt{\sum_i v_i^2}. \tag{3.9}$$

Identical shortest transfers of length L cost nL separately but $\sqrt{n}\,L$ in the unrestricted simultaneous product metric. A serial schedule implements the separate bill; it does not prove that the unrestricted product-space optimum is serial. A joint-FS formulation needs one-active-controller-at-a-time as an actual admissibility restriction.

3.3 Dependence on the phase reference

The lifted open-path Pancharatnam phase gives instead

$$\Gamma_{\mathrm{Pan}}(C) = 4 \arctan \frac{v-u}{2\epsilon} - 2\int_C \beta. \tag{3.10}$$

The minimizing path for each fixed pair is unchanged, but the minimized edge cost becomes $2H_\lambda(|v-u|/2)$ instead of (4.8). For $\epsilon = 1$, $\lambda = 0.6$, and intervals $[0, 0.9] \cup [0.95, 1.85]$, the outer–inner assignment minus the componentwise assignment has

$$\Delta_{\mathrm{fixed}} = 0.0733222076959\ldots, \qquad \Delta_{\mathrm{Pan}} = -0.0757727830054\ldots. \tag{3.11}$$

The winner reverses. All endpoint overlap phases have magnitude below π, so this is not an arctangent branch ambiguity. The altered winner is an assignment statement, not an assertion that its deformed curves make an embedded positive region. The fixed-width reference is physically part of the task rather than a removable gauge convention.

3.4 Composition and source constraints

The directed distance satisfies the triangle inequality and $d_\lambda(p,p) = 0$. Inserting a coincident source and sink at p leaves (3.6) unchanged: extend an old assignment by $p \to p$, while any pair $a \to p$, $p \to b$ in a new assignment can be replaced by $a \to b$ without increasing cost. The fixed reference sum is unchanged. Continuity gives the corresponding interval-coalescence value limit. A positive preparation fee per supplied probe would destroy that inventory-independent statement. Likewise, fixed balanced clusters sufficiently far apart have no winning intercluster edges, since $d_\lambda(u,v) = 2q\log(|u-v|/\epsilon) + O(1)$ diverges while intracluster assignments stay bounded. This separation claim is not uniform near $|\lambda| = 1$.

Let $\mathcal{C}_n(\Phi)$ be the infimum of the separate bill at prescribed real phase. Then

$$\mathcal{J}_\lambda = \inf_\Phi [\mathcal{C}_n(\Phi) + \lambda\Phi/2], \qquad \mathcal{C}_n(\Phi) \geq |W/\epsilon - \Phi/2|. \tag{3.12}$$

The second inequality is $\sum_i \int \alpha \geq |\sum_i \int \beta|$. It shows that the real-phase cost is not periodic. At $\epsilon = 1$, a capped loop of width 2π between $z = 1$ and $z = 2$ changes the real phase by -2π and costs $3\pi + 2\log 2$, while leaving terminal rays and $U(1)$ holonomy unchanged. Prescribing only a phase on the circle would instead minimize over all lifts. No source term $\lambda\Phi/2$ is single-valued on that circle.

Equation (3.12) is a partition of protocols, not strong Legendre duality. Source minimizers expose part of the fixed-phase frontier; other prescribed phases can require loops or intersections and need not correspond to positive-region area. On a smooth winning branch, $\partial_\lambda \mathcal{J}_\lambda = \mathrm{Area}_H(E_\lambda)$, with one-sided derivatives at transitions. For $\lambda > 1$, a rectangular loop of bottom width D at ϵ and top height Z has directed cost

$$D\left[\frac{1-\lambda}{\epsilon} + \frac{1+\lambda}{Z}\right] + 2\log(Z/\epsilon). \tag{3.13}$$

Choosing $Z > \epsilon(1+\lambda)/(\lambda-1)$ and then large D makes repeated loops unboundedly favorable. Reversing orientation treats $\lambda < -1$. At $\lambda = 1$, $\alpha - \beta \geq 0$ and cutoff translations attain $\mathcal{J}_1 = W/\epsilon$ on the field path domain. They have no positive-thickness exact-trace BV representative; thin positive collars approach the same value without attaining it. Thus critical sources lose coercivity and separate the path and positive-region domains. These control-domain facts reinforce, rather than enlarge, the subcritical scope of the actual-state correspondence.

4 Density-matrix geometry and spatial composition

Spatially specified state families contain more information than the union spectrum. We test whether their canonical square-root, Bures or Uhlmann geometry supplies the positive length, oriented area and composition required by the pressure problem. Each calculation specifies the regulator and the identification of Hilbert spaces. The results concern these constructions and comparison conventions, rather than all tasks using local algebras.

4.1 The canonical square-root section is Berry-flat

For a faithful normalized density matrix on a fixed Hilbert space, its canonical purification is the Hilbert–Schmidt vector $|\sqrt{\rho}\rangle\!\rangle$. Hermiticity gives

$$\langle\!\langle \sqrt{\rho} | \mathrm{d}\sqrt{\rho} \rangle\!\rangle = \mathrm{Tr}(\sqrt{\rho}\,\mathrm{d}\sqrt{\rho}) = \tfrac{1}{2}\mathrm{d}\,\mathrm{Tr}\,\rho = 0. \tag{4.1}$$

Its ordinary Berry connection and curvature therefore vanish. Moreover $\mathrm{Tr}(\sqrt{\rho}\sqrt{\sigma}) \geq 0$, strictly for faithful states, so canonical finite overlaps have no phase. Transporting the reference copy differently would add purification data.

In an eigenbasis of ρ, differentiating $(\sqrt{\rho})^2 = \rho$ gives

$$(\mathrm{d}\sqrt{\rho})_{ij} = \frac{(\mathrm{d}\rho)_{ij}}{\sqrt{p_i}+\sqrt{p_j}},$$
$$\mathrm{d}s^2_{\mathrm{can}} = \sum_{ij} \frac{|(\mathrm{d}\rho)_{ij}|^2}{(\sqrt{p_i}+\sqrt{p_j})^2}, \qquad \mathrm{d}s^2_B = \frac{1}{2}\sum_{ij} \frac{|(\mathrm{d}\rho)_{ij}|^2}{p_i+p_j}. \tag{4.2}$$

The first is the square-root pullback, one quarter of the conventionally normalized Wigner–Yanase metric [10]; it is generally not the Bures metric. On a unitary path generated by K, it is $2I_{\mathrm{WY}}(\rho, K)\mathrm{d}t^2$. For a varying pure state the canonical vector is $|\psi\rangle \otimes |\psi^*\rangle$: Berry phases cancel and the metric is twice the original pure-state FS metric. It must not be identified with the undoubled supplied probe.

For an actual product state, differentiating the tensor product of square roots and using (4.1) gives

$$\mathrm{d}s^2_{\mathrm{can},12} = \mathrm{d}s^2_{\mathrm{can},1} + \mathrm{d}s^2_{\mathrm{can},2}. \tag{4.3}$$

The Bures metric has the same product rule: its symmetric logarithmic derivative is the sum of the two factor derivatives, whose expectations vanish. Both joint speeds combine in quadrature, not as the sum of positive boundary lengths. Serial metering would again be an additional architecture.

4.2 An explicit single-interval comparison

The regulated vacuum interval can be mapped to an annulus of width $\mathcal{W} = 2\log(\ell/a_{\mathrm{UV}})+$ cutoff corrections. With fixed conformal boundary conditions, identify its fixed boundary-CFT energy basis and vary the width. The density family is [11]

$$\rho_{\mathcal{W}} = \frac{e^{-b_{\mathrm{an}}H_{ab}}}{Z(b_{\mathrm{an}})}, \qquad b_{\mathrm{an}} = 2\pi^2/\mathcal{W}, \qquad H_{ab}|j\rangle = (\Delta_j - c/24)|j\rangle. \tag{4.4}$$

This is a stated spectral comparison convention; translation of the interval is removed by the identification. All matrices commute, so the canonical and Bures metrics agree and the Uhlmann connection vanishes. Differentiating the thermal probabilities gives

$$\mathrm{d}s^2 = \tfrac{1}{4}\,\mathrm{Var}(H_{ab})\mathrm{d}b^2_{\mathrm{an}} = \frac{C_E}{4}\left(\frac{\mathrm{d}\mathcal{W}}{\mathcal{W}}\right)^2, \qquad C_E = \mathrm{Var}_\rho(-\log\rho). \tag{4.5}$$

In a vacuum-dominated closed channel, $\log Z = c\mathcal{W}/12 + O(1)$ and $C_E = c\mathcal{W}/6 + o(\mathcal{W})$, hence

$$\mathrm{d}s^2 \simeq \frac{c}{24\mathcal{W}}\mathrm{d}\mathcal{W}^2 = \frac{c}{12\log(\ell/a_{\mathrm{UV}})}\left(\frac{\mathrm{d}\ell}{\ell}\right)^2, \qquad g_{XX} = 0. \tag{4.6}$$

This rank-one metric has neither a translation direction nor a constant hyperbolic radial coefficient. In the oscillator annulus check $Z = \prod_{m\geq 1}(1-e^{-b_{\mathrm{an}}m})^{-1}$,

$$C_E = \sum_{m\geq 1} \frac{(b_{\mathrm{an}}m)^2 e^{-b_{\mathrm{an}}m}}{(1-e^{-b_{\mathrm{an}}m})^2} \simeq \mathcal{W}/6 - 1/2. \tag{4.7}$$

The values at $\mathcal{W} = 20, 40, 80$ are $2.83333\ldots$, $6.16666\ldots$, $12.83333\ldots$. This illustrates the leading $c = 1$ scale, not an exact spectrum for every CFT or an Einstein-dual test.

Other identifications may add eigenvector motion. The result does not assert that physical eigenvectors in every common regulator are width-independent; it says that the additional comparison transport must be specified. Acting with modular time on only one half of the purification also adds a reference clock while leaving ρ unchanged. Its metric component is $C_E \mathrm{d}t^2$; it is motion in the purification fiber, not an extra density-matrix coordinate.

4.3 Conformal covariance and comparison transport

There is a separate unregulated symmetry obstruction. Pulling the vacuum restriction on I back to a reference interval by its conformal algebra isomorphism gives a constant state family. Its state distinguishability metric is zero. A nontrivial geometry of embeddings or modular frames can nevertheless remain [12, 13].

At $I = (-1, 1)$, the stabilizer contains

$$g_t(x) = \frac{x \cosh t + \sinh t}{x \sinh t + \cosh t}, \qquad (\mathrm{d}u, \mathrm{d}v) \mapsto (e^{2t}\mathrm{d}u, e^{-2t}\mathrm{d}v). \tag{4.8}$$

Invariance of $a\mathrm{d}u^2 + 2b\mathrm{d}u\mathrm{d}v + d\mathrm{d}v^2$ forces $a = d = 0$; positivity then forces $b = 0$. Thus full conformal covariance alone cannot supply a nonzero positive metric on this orbit. The normalized modular-generator matrix makes its Lorentzian geometry explicit:

$$Q(X, r) = \frac{1}{r}\begin{pmatrix} X & r^2 - X^2 \\ 1 & -X \end{pmatrix}, \qquad Q^2 = 1, \qquad \tfrac{1}{2}\operatorname{Tr}(\mathrm{d}Q^2) = \frac{\mathrm{d}X^2 - \mathrm{d}r^2}{r^2}. \tag{4.9}$$

A fixed regulator or measuring apparatus can break this symmetry, but then that additional input belongs in the definition.

Uhlmann transport is not the flat canonical section and is not automatically modular-frame transport. Write an amplitude $w = \sqrt{\rho}\, V$ and $\mathcal{A}_U = \mathrm{d}V V^\dagger$. Its horizontality condition gives

$$\rho\mathcal{A}_U + \mathcal{A}_U\rho = [\mathrm{d}\sqrt{\rho}, \sqrt{\rho}], \qquad (\mathcal{A}_U)_{ij} = \frac{\sqrt{p_j} - \sqrt{p_i}}{p_i + p_j}(\mathrm{d}\sqrt{\rho})_{ij}. \tag{4.10}$$

For an isospectral family with instantaneous frame rotation ω, this becomes

$$(\mathcal{A}_U)_{ij} = \left[1 - \operatorname{sech}\frac{k_i - k_j}{2}\right]\omega_{ij}, \qquad k_i = -\log p_i. \tag{4.11}$$

The coefficient is 0.2 for weights $(0.8, 0.2)$ and $0.0202041\ldots$ for $(0.6, 0.4)$. Its actual spectral-gap dependence cannot be replaced by an unweighted modular rotation. General Uhlmann curvature need not vanish; it vanishes for the commuting family (4.4). A field-region comparison must derive its chosen connection before using it. Known holographic Uhlmann constructions concern a specified comparison and entanglement-wedge symplectic form [14, 15]; they do not by themselves establish the present scalar pressure minimum.

4.4 Composition of the actual disconnected reduction

For the actual union, generally $\rho_{A_1\cup A_2} \neq \rho_{A_1}\otimes\rho_{A_2}$. The canonical purification consequently does not factor by spatial components. The free massless fermion gives an exact continuum diagnostic: cross-interval correlations and bilocal modular couplings survive [16]. At a finite regulator, choose sites $\{0,1\}\cup\{4,5\}$ in the half-filled critical hopping chain. Its correlation matrix is

$$C_{ij} = \frac{\sin[\pi(i-j)/2]}{\pi(i-j)}, \qquad C_{ii} = 1/2, \qquad h = \log[(1-C)C^{-1}]. \tag{4.12}$$

The four-mode Gaussian state is proportional to $\exp(-\sum_{ij} h_{ij} c_i^\dagger c_j)$. For this state the cross-interval correlation norm is 0.1237366423, the modular-block norm 0.6169988919, and mutual information 0.07503131845 nats. Its canonical square root differs from the tensor product of the two marginal square roots by Hilbert–Schmidt norm 0.1799128558. These are a concrete nonfactorization check, not holographic large-c corrections.

None of these standard constructions derives, merely from disconnected labels, the separately additive length, freely chosen endpoint assignment and fixed-width oriented reference of section 3. Conversely, none rules out a different task using ρ_A and its spatial algebras with independently specified operations. The saving (1.3) uses precisely that extra component structure without postulating a density-matrix metric.

5 Replica channel weights and reflected entropy

Replica observables retain permutation, operator-product and spatial channel data beyond the union spectrum. We compare those data with the real orientation-dependent weight required by the pressure action. The first test concerns equally normalized inverse-twist identity pairings; the second concerns a direct reflected-entropy and Markov-gap source class. Optimized purifications, defects and more general replica networks require their own analysis.

5.1 The required orientation-dependent weight

For two intervals with cross-ratio η, the leading connected-minus-disconnected pressure action is

$$\frac{3}{c}\Delta_\lambda(\eta) = q\log\frac{1-\eta}{\eta} + \pi\lambda, \qquad \partial_\lambda \frac{3}{c}\Delta_\lambda(\eta)\Big|_0 = \pi. \tag{5.1}$$

The derivative is a branch statement on any fixed connected-channel window before minimization. More generally, if $r(M)$ is the number of reverse arcs in a directed assignment, (4.13) gives

$$\frac{3}{c}F_M = \text{common trace and endpoint terms} + q\sum_{e\in M}\log\Delta_e + \pi\lambda r(M) + o(1). \tag{5.2}$$

Thus a proposed replica saddle needs an orientation-sensitive real channel normalization, not only a rescaled twist dimension. Its gluing rules must derive that factor.

Consider the specified class of branches made from identity contractions of one equally normalized inverse-twist pair:

$$\langle \mathcal{T}_g(u)\mathcal{T}_{g^{-1}}(v)\rangle = N_g|u-v|^{-2\Delta_g}, \qquad N_g = N_{g^{-1}}. \tag{5.3}$$

Normalizing the twist fixes the identity OPE coefficient. Every complete pairing contains the same number of factors and endpoints, so local normalizations cancel from channel differences. Reversing endpoint order cannot produce an additional positive real weight. Equivalently, the crossing-related identity-channel difference is antisymmetric under $\eta \leftrightarrow 1-\eta$ and vanishes at $1/2$, whereas

$$\frac{3}{c}[\Delta_\lambda(\eta) + \Delta_\lambda(1-\eta)] = 2\pi\lambda. \tag{5.4}$$

Ordinary multi-interval replicas have nontrivial CFT-dependent correlators [17]; the gate concerns the stated equal-normalization identity-pairing ansatz, not every correlator or every possible saddle.

5.2 Reflected replica channel constants

For a specified bipartition ρ_{AB}, define

$$|\psi_m\rangle = \frac{|\rho_{AB}^{m/2}\rangle\!\rangle}{\sqrt{\operatorname{Tr}\rho_{AB}^m}}, \qquad \sigma_{AA^*}^{(m)} = \operatorname{Tr}_{BB^*}|\psi_m\rangle\langle\psi_m|,$$
$$\mathcal{R}_{m,k}(A:B) = \frac{1}{1-k}\log \operatorname{Tr}[(\sigma_{AA^*}^{(m)})^k]. \tag{5.5}$$

The replica index k here is unrelated to the number of spatial components. The AA^* reduction keeps eigenvector information relative to $A|B$; this is not a spectrum-only escort of ρ_{AB}.

In the reflected replica fusion, the exchanged permutation is nonidentity. Its dimensions and OPE constant are fixed together [18]:

$$h_{\text{ext}} = \frac{ck(m^2-1)}{24m}, \qquad h_{\text{int}} = \frac{2c(k^2-1)}{24k}, \qquad C_{k,m} = (2m)^{-4h_k}, \quad h_k = \frac{c}{24}(k-k^{-1}). \tag{5.6}$$

For nearby intervals, $x = 1-\eta \ll 1$, the controlled expansion is

$$\mathcal{R}_{m,k} = -\frac{c(k+1)}{6k}\log x + \frac{c(k+1)}{3k}\log(2m) + \cdots. \tag{5.7}$$

The finite constant is real and intrinsic to this observable. It cannot be varied while discarding the accompanying changes in dimensions and blocks. In the standard reflected limit, the connected holographic function is

$$S_R(A:B) = \frac{c}{3}\log\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}} + O(c^0). \tag{5.8}$$

This shape differs from (5.1). If B is trivial, (5.5) is zero because AA^* is the complete pure purification, so it also cannot directly reproduce a nonzero single-interval perimeter action. Taking a pure global bipartition instead reduces it to twice an ordinary Rényi entropy and returns to the spectral obstruction. Matching only the nearby-interval constant by choosing $m(\lambda)$ does not solve either problem.

5.3 A first-order reflected and Markov-gap obstruction

A natural attempt is to deform a baseline entropy using a fixed linear combination of reflected entropy and mutual information. On $1/2 < \eta < 1$, write

$$\mathsf{R}(\eta) = \frac{3}{c} S_R = \log \frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}, \qquad \mathsf{M}(\eta) = \frac{3}{c} I = \log \frac{\eta}{1-\eta}. \tag{5.9}$$

If the first source response after common local terms cancel were $a\mathsf{R}+b\mathsf{M}$, the target would require it to equal π throughout an open cross-ratio interval. But

$$\frac{\mathrm{d}}{\mathrm{d}\eta}(a\mathsf{R} + b\mathsf{M}) = \frac{a\sqrt{\eta}+b}{\eta(1-\eta)}. \tag{5.10}$$

Vanishing throughout such an interval forces $a = b = 0$, which yields zero rather than π. A branch-specific constant would insert the missing datum by hand. In particular the Markov gap obeys

$$\mathsf{R} - \mathsf{M} = 2\log(1+\eta^{-1/2}), \qquad (\mathsf{R}-\mathsf{M})' = -\frac{1}{\eta(1+\sqrt{\eta})} \neq 0. \tag{5.11}$$

It tends to $\log 4$ as $\eta \to 1$ but is not that constant at general cross-ratio: its values at 0.55 and 0.95 are 1.707468256 . . . and 1.412105441 . . .. Topology-sensitive Markov-gap and recovery bounds are inequalities [19]; replacing this function by a lower bound would change the observable. The calculation excludes this first-order source class, not every generalized reflected Rényi combination.

5.4 Additional replica and boundary data

Replica tension is not fixed-background pressure. A cosmic brane has $T_k = (k-1)/(4kG)$ and varies jointly with the gravitational metric [20]. If the metric were held fixed and only a constant multiple of length varied, the embedding equation would remain geodesic curvature zero, not λ/R. An equivalent fixed-background pressure description would need an additional proved map with the correct cutoff and anchors. Finite-replica reflected computations likewise change the ambient geometry, not only an isolated OPE constant.

A physical conformal boundary can supply a genuine finite boundary entropy. In the elementary AdS/BCFT construction,

$$\log g = \frac{c}{6}\,\mathrm{artanh}(RT), \tag{5.12}$$

where the boundary state or tension is additional physical input [21]. Such observables are not excluded by the identity-twist gate. Choosing a defect or $g(\lambda)$ after matching the desired channel penalty would, however, define a different task unless the boundary operation was independently specified.

Similarly, a finite Euler-characteristic counterterm can remove the $\pi\lambda r(M)$ term in (5.2), restoring the ordinary $\eta = 1/2$ competition. It changes the positive-area target; common endpoint terms cannot accomplish that change. These distinctions isolate the missing requirements of a future replica candidate: derive its channel weight, source-zero branch response and cross-ratio dependence from the same field definition. They do not assert that no such construction can exist.

6 Reproduction and status

The common package retains the complement, interval-entropy, supplied-probe, density and replica producers. They were rerun as part of the 19-entry Python suite in the revision execution record. These finite tests have distinct parameter domains and tolerances; there is no shared numerical error certificate. The density calculation includes explicit finite-value checks in the revision. Mathematical claims remain restricted to the hypotheses stated in their respective sections.

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